HOMOLOGY OF A LEAVITT PATH ALGEBRA VIA ANICK'S RESOLUTION

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February 26, 2014

Abstract

The aim of this paper is to calculate the homology of a Leavitt path algebra via Anick's resolution. We show that all homology (in positive degrees) of a Leavitt path algebra is equal to zero.

Introduction

Anick's resolution was obtained by David J. Anick in 1986 [3]. This is a resolution for a field $\Bbbk$ considered as an A-module, where A is an associative augmented algebra over $\Bbbk$. This resolution reflects the combinatorial properties of A because it is based on the Composition–Diamond Lemma [5, 4]; i.e., Anick defined the set of n-chains via the leading terms of a Gröbner – Shirshov basis [13, 15, 6] (Anick called it the set of obstructions), and differentials are defined inductively via $\Bbbk$-module splitting maps, leading terms, and normal forms of words.

Later Yuji Kobayashi [11] obtained the resolution for a monoid algebra presented by a complete rewriting system. He constructed an effective free acyclic resolution of modules over the algebra of the monoid whose chains are given by paths in the graph of reductions. These chains are a particular case of chains defined by Anick [3], and differentials have "Anick's spirit", i.e., the differentials are described inductively via contracting homotopy, leading terms, and normal forms. Further Philippe Malbos [12] constructed a free acyclic resolution in the same spirit for $R\mathscr{C}$ as a $\mathscr{C}$-bimodule over a commutative ring R, where $\mathscr{C}$ is a small category endowed with a convergent presentation. The resolution is constructed using additive Kan extension of Anick's antichains generated by the set of normal forms. This construction can be adapted to the construction of the analogous resolution for internal monoids in a monoidal category admitting a finite convergent presentation. Malbos also showed (using the resolution) that if a small category admits a finite convergent presentation then its Hochschild–Mitchell homology is of finite type in all degrees.

Anick's resolution has cumbersome differentials, which are hard to compute. Farkas [9] obtained formulas which described Anick's differentials easier but then Vladimir Dotsenko and Anton Khoroshkin [8] showed that Farkas's formulas for the Batalin–Vilkovisky algebra are false, i.e., Farkas's formulas are in general incorrect (see [8, 2.5]). They also give an answer to Malbos's question [12] whether Anick's resolution can be extended to the case of operads.

Sköldberg [14] obtained very interesting connections between Anick's resolution and Discrete Morse Theory, see [10] for further interesting details.

In this paper, we use Anick's resolution for calculating the homology of a Leavitt path algebra. This algebra was introduced in [1] as an algebraic analog of graph Cuntz–Kreiger C^*-algebra.

1 Anick's Resolution and General Remarks

Throughout this paper, $\Bbbk$ denotes any field and A is an associative $\Bbbk$-algebra with unity and augmentation; i.e., a $\Bbbk$-algebra homomorphism $\varepsilon : A \to \Bbbk$. Let X be a set of generators for A. Suppose that $\le$ is a well-ordering on the free monoid generated by X. For instance, given a fixed order on the letters, one may order words "length-lexicographically" by first ordering by length and then comparing words of the same length by checking which of them occurs earlier in the dictionary. Denote by $\Bbbk\langle X\rangle$ the free associative $\Bbbk$-algebra with unity on X. There is a canonical surjection $f : \Bbbk\langle X\rangle \to A$ once X is chosen, in other words, we get $A \cong \Bbbk\langle X\rangle/\ker(f)$

Let GSB_A be a Gröbner–Shirshov basis for A. Denote by $\mathfrak{V}$ the set of the leading terms in GSB_A and let $\mathfrak{M} = \mathrm{Irr}(\ker(f))$ be the set of irreducible words (not containing the leading monomials of relators as subwords) or $\Bbbk$-basis for A (see CD-Lemma [5, 4]). Following Anick [3], $\mathfrak{V}$ is called the set of obstructions (antichains) for $\mathfrak{M}$. For $n \ge 1$, $\upsilon = x_{i_1} \cdots x_{i_t} \in X^*$ is an *n-prechain* iff there exist $a_j, b_j \in \mathbb{Z}$, $1 \le j \le n$, satisfying

1. $1 = a_1 < a_2 \le b_1 < a_3 \le b_2 < \ldots < a_n \le b_{n-1} < b_n = t$ and,

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2. $x_{i_{a_j}} \cdots x_{i_{b_j}} \in \mathfrak{V}$ for $1 \leq j \leq n$.

A n-prechain $x_{i_1} \cdots x_{i_t}$ is an *n-chain* iff the integers $\{a_j, b_j\}$ can be chosen so as to satisfy

3. $x_{i_1} \cdots x_{i_s}$ is not an m-prechain for any $s < b_m$, $1 \leq m \leq n$.

As in [3], we say that the elements of X are 0-chains, the elements of $\mathfrak{V}$ are 1-chains, and denote the set of n-chains by $\mathfrak{V}^{(n)}$.

Given a set X, denote by X^* the free monoid generated by X and, following [3], for any subset $U \subseteq X^*$, let $U\Bbbk = \mathrm{Span}_{\Bbbk} U$ denote the $\Bbbk$-submodule of $\Bbbk\langle X\rangle$ spanned by U. From [Lemma 1.1][3] it follows that $U\Bbbk \otimes_{\Bbbk} A$ has a basis $\widehat{U} = \{u \otimes f(x) | u \in U,\ x \in \mathfrak{M}\}$, and we can define a partial order on $\widehat{U}$ by writing $u \otimes x < u' \otimes x'$ iff $ux < u'x'$ in X^*. When U is the set of n-chains then the case $ux = u'x'$ is impossible by property (c). When $\omega = \sum\limits_{j=1}^{n} c_j (u_j \otimes v_j) \in U\Bbbk \otimes_{\Bbbk} A$ with $c_j \in \Bbbk - (0)$, $u_j \in U$, $v_j \in \mathfrak{M}$, we say that $u_1 \otimes v_1$ is the *leading term* of ω iff $q = 1$ or $u_1 \otimes v_1 > u_j \otimes v_j$ for all $j \neq 1$. Such a situation will also we denoted by $\mathrm{HT}(\omega) = u_1 v_1$, denote also $\mathrm{HT}_c(\omega)$ the leading term of ω with coefficients.

In [3], Anick proved the following Theorem, which we rewrite in terms of Gröbner–Shirshov bases and the Composition–Diamond Lemma; see [7] for a discussion of Anick's resolution in terms of Gröbner–Shirshov bases.

THEOREM 1.1. *Let A be an associative augmented $\Bbbk$-algebra generated as a $\Bbbk$-algebra by a set X and let $\leq$ be a well-ordering on the free monoid generated by X. Suppose that GSB_A is a Gröbner–Shirshov basis for A, $\mathfrak{V}$ is the set of the leading terms (1-chains) of GSB_A, and $\mathfrak{V}^{(n)}$ is the set of n-chains. Then there is a free A-resolution of $\Bbbk$,*

$$0 \leftarrow \Bbbk \xleftarrow{\varepsilon} A \xleftarrow{d_0} X\Bbbk \otimes_{\Bbbk} A \xleftarrow{d_1} \mathfrak{V}\Bbbk \otimes_{\Bbbk} A \xleftarrow{d_2} \mathfrak{V}^{(2)}\Bbbk \otimes_{\Bbbk} A \xleftarrow{d_3} \mathfrak{V}^{(3)}\Bbbk \otimes_{\Bbbk} A \leftarrow$$

where

$$d_0(x \otimes 1) = x - \varepsilon(x),$$

for $x \in X$ and if $n \geq 1$ then

$$d_n(x_{\iota_1} \cdots x_{\iota_{b_n}} \otimes 1) = x_{\iota_1} \cdots x_{\iota b_{n-1}} \otimes x_{\iota_{b_{n-1}+1}} \cdots x_{\iota_{b_n}} + \vartheta,$$

where $\mathrm{HT}(\vartheta) < x_{\iota_1} \cdots x_{\iota_{b_n}}$ if $\omega \neq 0$.

REMARK 1.1. Anick [1] defined differentials for any $(n+1)$-chain, $n \geq 0$, $x_{\iota_{a_1}} \cdots x_{\iota_{b_{n+1}}} \in \mathfrak{V}^{(n+1)}$ by

$$d_{n+1}(x_{\iota_{a_1}} \cdots x_{\iota_{b_{n+1}}} \otimes 1) = \\ = x_{\iota_{a_1}} \cdots x_{\iota_{b_n}} \otimes x_{\iota_{b_n+1}} \cdots x_{\iota_{b_{n+1}}} - i_n d_n(x_{\iota_{a_1}} \cdots x_{\iota_{b_n}} \otimes x_{\iota_{b_n+1}} \cdots x_{\iota_{b_{n+1}}}).$$

Here $x_{\iota_{a_1}} \cdots x_{\iota_{b_n}} \in \mathfrak{V}^{(n)}$ is an n-chain and $i_n : \ker(d_{n-1}) \to \mathfrak{V}^{(n)}\Bbbk \otimes_{\Bbbk} A$ is a $\Bbbk$-module splitting map (i.e., $d_n i_n = \mathrm{id}_{\ker(d_{n-1})}$). From [p.645][3] it follows that we can describe these differentials of the resolution as follows: First of all, define a function $h_n : X^*\Bbbk \to \mathfrak{V}^{(n)}\Bbbk \otimes A$ by

$$h_n(w) = \begin{cases} v_n \otimes u, \text{ if there exist unique } v_n \in \mathfrak{V}^{(n)} \text{ and } u \in A \text{ such that } w = v_n u; \\ 0, \text{ otherwise.} \end{cases}$$

Further, define $\mu : X^*\Bbbk \otimes_{\Bbbk} X^*\Bbbk \to X^*\Bbbk$ by the equality $\mu(u \otimes v) = uv$. Then the differentials of Anick's resolution can be described by the formula

$$d_{n+1} = h_n \mu - \sum_{m \geq 0} h_n \mathrm{HT}_c(\omega^{(m)}). \tag{1}$$

Here $\omega^{(0)} = \omega = d_n h_n \mu$ and $\omega^{(j)} = \omega^{(j-1)} - c_j d_n h_n \mathrm{HT}_c \omega^{(j-1)}$ for $j \geq 1$, here $c_j \in \Bbbk$ are such that

$$c_j \mathrm{HT}_c d_n h_n \mathrm{HT}_c(\omega^{(j-1)}) = \mathrm{HT}_c \omega^{(j-1)}.$$

The main difficulty in Anick's resolution is finding $\mathrm{HT}(\omega^{(m)})$ since there are always two possibilities at mth step:

$$\mathrm{HT}(\omega^{(m)}) \in \{\mathrm{HT}(\omega^{(m-1)} - h_n \mathrm{HT}_c(\omega^{(m-1)})), \\ \mathrm{HT}(d_n h_n \mathrm{HT}(\omega^{(m-1)}) - h_n \mathrm{HT}_c d_n h_n \mathrm{HT}(\omega^{(m-1)}))\},$$

for fixed $n \geq 0$.

In this article, we show that there exists a algebra satisfying the following "nice" condition:

$$\mathrm{HT}(\omega^{(m)}) = \mathrm{HT}(\omega^{(m-1)} - h_n \mathrm{HT}_c(\omega^{(m-1)})) \tag{2}$$

for any $n \geq 0$, $m \geq 1$. In this case, the differentials in Anick's resolution are described very easily. Indeed, (1.1) immediately implies

$$d_{n+1} = h_n \mu - \sum_{m \geq 1} h_n m_c^{th} d_n h_n \mu. \tag{3}$$

Here m_c^{th} stands for the mth term of $d_n h_n \mu$ with coefficients.

LEMMA 1.1. *Suppose that* $\omega = \sum_{t=1}^{T} a_t \varphi_t \in \ker(d_{n-1})$, *where all* $a_t \in \Bbbk$ *and all* $\varphi_t \in \mathfrak{V}^{(n-1)} k \otimes A$ *are such that for all* t *with* $1 \leq t \leq T$ *we have:*

1. $h_n \mathrm{HT} \varphi_t \neq 0$,
2. $d_n h_n \mathrm{HT} \varphi_t = \varphi_t + \psi_t$, *where* ψ_t *is such that* $\max \deg \psi_t < \max \deg \varphi_t$ *for any* $1 \leq t \leq T$, *then*

$$i_n(\omega) = \sum_{t=1}^{T} a_t h_n \mathrm{HT} \varphi_t + i_n(\omega')$$

where $\omega' = -\sum_{t=1}^{T} a_t \psi_t$

PROOF. Indeed, let $\mathrm{HT}(\omega) = \mathrm{HT}(\varphi_{t_p})$ for some $1 \leq t_p \leq T$. Then $\omega^{(1)} = \omega - a_{t_p} d_n h_n \mathrm{HT}(\varphi_{t_p}) = \sum_{t=1, t \neq t_p}^{T} a_t \varphi_t - a_{t_p} \psi_{t_p}$, but since $\max \deg \psi_{t_p} < \max \deg \varphi_t$, we can assume that $\mathrm{HT}(\omega^{(1)}) = \mathrm{HT}(\varphi_{t_q})$ for some $1 \leq t_q \neq t_p \leq T$, and we get $\omega^{(2)} = \sum_{t=1, t \neq t_p, t \neq t_q} a_t \varphi_t - a_{t_p} \psi_{t_p} - a_{t_q} \psi_{t_q}$. Thus, after T steps we have $\omega^{(T)} = -\sum_{t=1}^{T} a_t \psi_t$ and $i_n(\omega) = \sum_{t=1}^{T} a_t h_n \mathrm{HT} \varphi_t + i_n(\omega^{(T)})$, q.e.d.

□

2 Homology of a Leavitt Path Algebra

A directed graph $\Gamma = (V, E, \mathrm{dom}, \mathrm{cod})$ consists of two sets V and E, called vertices and edges respectively, and two maps $\mathrm{dom}, \mathrm{cod} : E \to V$ called the domain and the codomain (of an edge) respectively. A graph is called *row-finite* if $|\mathrm{dom}^{-1}(v)| < \infty$ for all vertices $v \in V$. A vertex v for which $\mathrm{dom}^{-1}(v)$ is empty is called a *sink*.

DEFINITION 2.1. *Let* Γ *be a row-finite graph. The Leavitt path* $\Bbbk$*-algebra* $L(\Gamma)$ *is the* $\Bbbk$*-algebra presented by the set of generators* $\{v, v \in V\}$, $\{e, e^* | e \in E\}$ *and the following set of relations:*

1. $v_i v_j = \delta_{i,j} v_i$ *for all* $v_i, v_j \in V$,
2. $\mathrm{dom}(e)e = e\mathrm{cod}(e) = e$, $\mathrm{cod}(e)e^* = e^*\mathrm{dom}(e) = e^*$, *for all* $e \in E$,
3. $a^* b = \delta_{a,b} \mathrm{cod}(a)$, *for all* $a, b \in E$,
4. $v = \sum_{\mathrm{dom}(e) = v} ee^*$ *for any vertex* $v \in V \setminus \{\mathrm{sinks}\}$.

Condition 4 can be reformulated as follows: Let $e_v^1 > e_v^2 > \ldots > e_v^\ell$ be all edges that originate from v (we put $e_v^1 = e_v$ for brevity). Then we get

4'. $e_v e_v^* = \mathrm{dom}(e_v) - \sum_{r=2}^{\ell} e_v^r e_v^{r*}$.

The following proposition was proved in [2] but not all equations were given. We expose the "complete" version of the Gröbner–Shirshov basis for a Leavitt path algebra.

PROPOSITION 2.1. *A Leavitt path algebra admits a Gröbner — Shirshov basis described by following equalities:*

1. $v_i v_j = \delta_{i,j} v_i$,
2. $\mathrm{dom}(a)b = \delta_{\mathrm{dom}(a),\mathrm{dom}(b)} b$, $a\,\mathrm{cod}(b) = \delta_{\mathrm{cod}(a),\mathrm{cod}(b)} a$,
3. $\mathrm{cod}(a)b^* = \delta_{\mathrm{cod}(a),\mathrm{cod}(b)} b^*$, $a^*\mathrm{dom}(b) = \delta_{\mathrm{dom}(a),\mathrm{dom}(b)} a^*$,
4. $a^* b = \delta_{a,b} \mathrm{cod}(a)$, $ab = \delta_{\mathrm{cod}(a),\mathrm{dom}(b)} ab$, $a^* b^* = \delta_{\mathrm{dom}(a),\mathrm{cod}(b)} a^* b^*$,
5. $e_v e_v^* = \mathrm{dom}(e_v) - \sum_{r=2}^{\ell} e_v^r e_v^{r*}$,
6. $ab^* = \delta_{\mathrm{cod}(a),\mathrm{cod}(b)} ab^*$, *iff* $a \neq b$ *and* $\mathrm{dom}(a) \neq \mathrm{dom}(b)$.

Using this proposition, it is not hard to prove

PROPOSITION 2.2. *Anick's* n*-chains* $\mathfrak{V}^n_{L(\Gamma)}$ *for a Leavitt path algebra for any* $n \geq 0$ *can be described as sets of the form* $\{\xi_1 \cdots \xi_{n+1}\}$ *such that:*

- *if* $\xi_i = v$ *then* $\xi_{i-1}, \xi_{i+1} \in V \cup E \cup E^*$,

- *if* $\xi_i = e_v$ *then* $\xi_{i-1} \in V \cup E^* \cup \{a \in E : \operatorname{cod}(a) \neq v\}$, $\xi_{i+1} \in V \cup \{e_v^*\} \cup \{a \in E : \operatorname{dom}(a) \neq \operatorname{cod}(e_v)\} \cup \{a^* \in E^* : \operatorname{cod}(a) \neq \operatorname{cod}(e_v),\ \operatorname{dom}(a) \neq v\}$,
- *if* $\xi_i = e_v^*$ *then* $\xi_{i-1} \in V \cup \{e_v\} \cup \{a \in E : \operatorname{cod}(a) \neq \operatorname{cod}(e_v), \operatorname{dom}(a) \neq v\} \cup \{a^* \in E^* : \operatorname{dom}(a) \neq \operatorname{cod}(e_v)\}$, $\xi_{i+1} \in V \cup E \cup \{a^* \in E^* : \operatorname{cod}(a) \neq v\}$,
- *if* $\xi_i = e_v^r$, *here* $r > 1$ *then* $\xi_{i-1} \in V \cup \{a \in E : \operatorname{cod}(a) \neq v\} \cup E^*$, $\xi_{i+1} \in V \cup \{a \in E : \operatorname{cod}(e_v^r) \neq \operatorname{dom}(a)\} \cup \{a^* \in E^* : \operatorname{cod}(a) \neq \operatorname{cod}(e_v^r), a \neq e_v^r,\ \operatorname{dom}(a) \neq v\}$,
- *if* $\xi_i = e_v^{r*}$, *here* $r > 1$ *then* $\xi_{i-1} \in V \cup \{a \in E : a \neq e^r,\ \operatorname{cod}(a) \neq \operatorname{cod}(e_v^r),\ \operatorname{dom}(a) \neq v\} \cup \{a^* \in E^* : \operatorname{dom}(a) \neq \operatorname{cod}(e_v^r)\}$, $\xi_{i+1} \in V \cup E \cup \{a^* \in E^* : \operatorname{cod}(a) \neq v\}$.

PROOF follows from the fact that all chains overlap by a letter and Proposition 2.1.

□

We will need the following Lemma, where $\widetilde{f}$ stand for the lower term of a polynomial f and by f_j designates $f(\xi_j\xi_{j+1})$.

LEMMA 2.1. *Let* $\{\xi_1 \cdots \xi_{n+1}\} \in \mathfrak{V}^{(n)}$ *be an Anick* n*-chain for a Leavitt path algebra. Then*

$$\sum_{s=1}^{n}\sum_{t=1}^{n-1}(-1)^{n-s-1}(-1)^{n-t-1}\xi_1 \cdots \widetilde{f}_s \cdots \widetilde{f}_t \cdots \xi_{n+1} = 0. \tag{4}$$

PROOF. For convenience, denote the form on left-hand side of (4) by $\ddot{\omega} = \sum\limits_{s=1}^{n}\sum\limits_{t=1}^{n-1} a_{st}$ and observe that $a_{mm} = -a_{m+1,m}$ for each $1 \leq m < n$. Indeed,

$$a_{mm} = (-1)^{2n-2m-2}\xi_1 \cdots \xi_{m-1}\widetilde{f}(\widetilde{f}(\xi_m\xi_{m+1})\xi_{m+2})\xi_{m+3} \cdots \xi_{n+1};$$

and

$$a_{m+1,m} = (-1)^{2n-2m-3}\xi_1 \cdots \xi_{m-1}\widetilde{f}(\xi_m\widetilde{f}(\xi_{m+1}\xi_{m+2}))\xi_{m+3} \cdots \xi_{n+1}.$$

The construction of Gröbner–Shirshov bases implies the equation $f(\xi f(\xi'\xi'')) = f(f(\xi\xi')\xi'')$ since 1-chains overlap by a letter. Further, $a_{st} = -a_{t+1,s}$ for $s \neq t$. Really,

$$a_{st} = (-1)^{2n-(s+t)-2}\xi_1 \cdots \xi_{s-1}\widetilde{f}_s(\xi_s\xi_{s+1})\xi_{s+2} \cdots \xi_t\widetilde{f}_t(\xi_{t+1}\xi_{t+2})\xi_{t+3} \cdots \xi_{n+1};$$

and

$$a_{t+1,s} = (-1)^{2n-(s+t+1)-2}\xi_1 \cdots \xi_{s-1}\widetilde{f}_s(\xi_s\xi_{s+1})\xi_{s+2} \cdots \xi_t\widetilde{f}_t(\xi_{t+1}\xi_{t+2})\xi_{t+3} \cdots \xi_{n+1}.$$

□

THEOREM 2.1. *Let* $L(\Gamma)$ *be the Leavitt path algebra corresponding to a row-finite directed graph* Γ*. Then Anick's resolution for* $L(\Gamma)$*-module* $\Bbbk$ *is given by the (exact) chain complex*

$$0 \leftarrow \Bbbk \xleftarrow{\varepsilon} L(\Gamma) \xleftarrow{d_0} \operatorname{Span}_{\Bbbk}(V \cup E \cup E^*) \otimes_{\Bbbk} L(\Gamma) \xleftarrow{d_1}$$
$$\xleftarrow{d_1} \operatorname{Span}_{\Bbbk}\mathfrak{V}_{L(\Gamma)} \otimes_{\Bbbk} L(\Gamma) \xleftarrow{d_2} \operatorname{Span}_{\Bbbk}\mathfrak{V}^{(2)}_{L(\Gamma)} \otimes_{\Bbbk} L(\Gamma) \leftarrow \ldots.$$

Here the differentials are defined by the formulas

$$d_0(\xi \otimes 1) = \xi, \qquad \varepsilon(\xi) = 0, \qquad \text{for any } \xi \in V \cup E \cup E^*,$$

and

$$d_n(\xi_1 \cdots \xi_{n-1}e_v e_v^* \otimes 1) =$$
$$= \sum_{r=1}^{\ell}\xi_1 \cdots \xi_{n-1}e_v^r \otimes e_v^{r*} + \sum_{j=1}^{n}(-1)^{n-j-1}\xi_1 \cdots \widetilde{f}_j \cdots \xi_{n-1}e_v e_v^* \otimes 1 \tag{5}$$

for each $n > 0$*, and if* $\xi_n \neq e_v$, $\xi_{n+1} \neq e_v^*$ *then*

$$d_n(\xi_1 \cdots \xi_{n+1} \otimes 1) = \xi_1 \cdots \xi_n \otimes \xi_{n+1} + \sum_{j=1}^{n}(-1)^{n-j-1}\xi_1 \cdots \widetilde{f}_j \cdots \xi_{n+1} \otimes 1. \tag{6}$$

PROOF is carried out by induction on n. Suppose that (5) and (6) hold for some $n > 0$. Consider $d_{n+1}(\xi_1 \cdots \xi_{n+2} \otimes 1)$. We have $d_{n+1}(\xi_1 \cdots \xi_{n+2} \otimes 1) = \xi_1 \cdots \xi_{n+1} \otimes \xi_{n+2} - i_n d_n(\xi_1 \cdots \xi_{n+1} \otimes \xi_{n+2})$. Let $\omega = d_n(\xi_1 \cdots \xi_{n+1} \otimes \xi_{n+2}) = \varphi_{n+1} + \sum\limits_{j=1}^{n}(-1)^{n-j-1}\varphi_j$, where

$$\varphi_{n+1} = \begin{cases} \sum\limits_{r=1}^{\ell}\xi_1 \cdots \xi_{n-1}e_v^r \otimes f(e_v^{r*}\xi_{n+2}) \text{ iff } \xi_n = e_v\ \xi_{n+1} = e_v^*; \\ \xi_1 \cdots \xi_n \otimes f(\xi_{n+1}\xi_{n+2}) \text{ otherwise} \end{cases}$$

$$\varphi_j = \xi_1 \cdots \widetilde{f_j} \cdots \xi_{n+1} \otimes \xi_{n+2}, \text{ for } 1 \le j \le n.$$

So, let us check the conditions of Lemma 1.1 for φ_j if $1 \le j \le n$. We have

$$(-1)^{n-j-1} h_n \mathrm{HT}(\varphi_j) = (-1)^{n-j-1} \xi_1 \cdots \widetilde{f_j} \cdots \xi_{n+2} \otimes 1, \qquad 1 \le j \le n \tag{7}$$

Using (6), we get

$$\begin{aligned}(-1)^{n-j-1} d_n h_n \mathrm{HT}\varphi_j &= (-1)^{n-j-1} d_n(\xi_1 \cdots \widetilde{f_j} \cdots \xi_{n+2} \otimes 1) = \\ &= (-1)^{n-j-1} \xi_1 \cdots \widetilde{f_j} \cdots \xi_{n+1} \otimes \xi_{n+2} + \\ &+ (-1)^{n-j-1} \sum_{p=1}^{n} (-1)^{n-p-1} \xi_1 \cdots \widetilde{f_j} \cdots \widetilde{f_p} \cdots \xi_{n+2} \otimes 1 = \\ &= \varphi_j + \psi_j;\end{aligned}$$

i.e.,

$$\psi_j = (-1)^{n-j-1} \sum_{p=1}^{n} (-1)^{n-p-1} \xi_1 \cdots \widetilde{f_j} \cdots \widetilde{f_p} \cdots \xi_{n+2} \otimes 1, \qquad 1 \le j \le n \tag{8}$$

we see that $\max \deg \psi_j = n$, for any $1 \le j \le n$.

1. Suppose that $\xi_n = e_v$ and $\xi_{n+1} = e_v^*$. Then

$$\omega = \sum_{r=1}^{\ell} \xi_1 \cdots \xi_{n-1} e_v^r \otimes f(e_v^{r*} \xi_{n+2}) + \sum_{j=1}^{n} (-1)^{n-j-1} \xi_1 \cdots \widetilde{f_j} \cdots \xi_{n-1} e_v e_v^* \otimes \xi_{n+2},$$

and we have

$$f(e_v^{r*} \xi_{n+2}) = \begin{cases} e_v^{r*}, \text{ iff } \xi_{n+2} = \mathrm{dom}(e_v) \text{ for all } r \ge 1, \\ \mathrm{cod}(e_v^r), \text{ if } \xi_{n+2} = e_v^r, \text{ for } r \ge 1, \\ 0, \text{ otherwise }. \end{cases}$$

(a) Let $f(e_v^{r*} \xi_{n+2}) = e_v^{r*}$. Then

$$\begin{aligned}\omega &= \sum_{r=1}^{\ell} \xi_1 \cdots \xi_{n-1} e_v^r \otimes e_v^{r*} + \sum_{j=1}^{n} (-1)^{n-j-1} \xi_1 \cdots \widetilde{f_j} \cdots \xi_{n-1} e_v e_v^* \otimes \xi_{n+2} = \\ &= \varphi_{n+1} + \sum_{j=1}^{n} (-1)^{n-j-1} \varphi_j,\end{aligned}$$

where

$$h_n \mathrm{HT}(\varphi_{n+1}) = \xi_1 \cdots \xi_{n-1} e_v e_v^* \otimes 1. \tag{9}$$

Therefore, by (5) we obtain

$$\begin{aligned}d_n h_n \mathrm{HT}(\varphi_{n+1}) &= d_n(\xi_1 \cdots \xi_{n-1} e_v e_v^* \otimes 1) = \\ &= \sum_{r=1}^{\ell} \xi_1 \cdots \xi_{n-1} e_v^r \otimes e_v^{r*} + \sum_{p=1}^{n} (-1)^{n-p-1} \xi_1 \cdots \widetilde{f_p} \cdots \xi_{n-1} e_v e_v^* \otimes 1 = \\ &= \varphi_{n+1} + \psi_{n+1},\end{aligned}$$

and we have $\max \deg \psi_{n+1} = n$,

$$\psi_{n+1} = \sum_{p=1}^{n} (-1)^{n-p-1} \xi_1 \cdots \widetilde{f_p} \cdots \xi_{n-1} e_v e_v^* \otimes 1. \tag{10}$$

By (7), (9), and Lemma 1.1, we have

$$\begin{aligned}i_n(\omega) &= \sum_{j=1}^{n} (-1)^{n-j-1} \xi_1 \cdots \widetilde{f_j} \cdots \xi_{n-1} e_v e_v^* \xi_{n+2} \otimes 1 + \\ &+ (-1)^{n-(n+1)-1} \xi_1 \cdots \xi_{n-1} e_v \widetilde{f_{n+1}}(e_v^* \xi_{n+2}) \otimes 1 + i_n(\omega') = \\ &- \sum_{j=1}^{n+1} (-1)^{n-j} \xi_1 \cdots \widetilde{f_j} \cdots \xi_{n-1} e_v e_v^* \xi_{n+2} \otimes 1 + i_n(\omega').\end{aligned} \tag{11}$$

Combining Lemma 1.1, (8), and (10), we infer

$$\omega' = -\sum_{j=1}^{n}\psi_j - \psi_{n+1} = -\sum_{p=1}^{n}\sum_{j=1}^{n}(-1)^{n-p-1}(-1)^{n-j-1}\xi_1\cdots\widetilde{f_p}\cdots\widetilde{f_j}\cdots\xi_{n+2}\otimes 1-$$
$$-\sum_{p=1}^{n}(-1)^{n-p-1}\xi_1\cdots\widetilde{f_p}\cdots\xi_{n-1}e_v e_v^*\otimes 1 =$$
$$= -\sum_{p=1}^{n}\sum_{j=1}^{n}(-1)^{n-p-1}(-1)^{n-j-1}\xi_1\cdots\widetilde{f_p}\cdots\widetilde{f_j}\cdots\xi_{n+2}\otimes 1-$$
$$-(-1)^{n-(n+1)-1}\sum_{p=1}^{n}(-1)^{n-p-1}\xi_1\cdots\widetilde{f_p}\cdots\xi_{n-1}e_v\widetilde{f_{n+1}}(e_v^*\xi_{n+2})\otimes 1 =$$
$$= -\sum_{p=1}^{n}\sum_{j=1}^{n+1}(-1)^{n-p-1}(-1)^{n-j-1}\xi_1\cdots\widetilde{f_p}\cdots\widetilde{f_j}\cdots\xi_{n+2}\otimes 1.$$

Now, using Lemma 2.1, we get $\omega' = 0$.

(b) Suppose that $f(e_v^{r*}\xi_{n+2}) = \mathrm{cod}(e_v^r)$ for some $1 \le r \le \ell$. Then

$$\omega = \xi_1\cdots\xi_{n-1}e_v^r\otimes\mathrm{cod}(e_v^r) + \sum_{j=1}^{n}(-1)^{n-j-1}\xi_1\cdots\widetilde{f_j}\cdots\xi_{n-1}e_v^r e_v^{r*}\otimes e_v^r =$$
$$= \varphi_{n+1} + \sum_{j=1}^{n}(-1)^{n-j-1}\varphi_j.$$

Consequently,

$$h_n\mathrm{HT}(\varphi_{n+1}) = \xi_1\cdots\xi_{n-1}e_v^r\mathrm{cod}(e_v^r)\otimes 1, \tag{12}$$

and by (6) we obtain

$$d_n h_n\mathrm{HT}(\varphi_{n+1}) = d_n(\xi_1\cdots\xi_{n-1}e_v^r\mathrm{cod}(e_v^r)\otimes 1) = \xi_1\cdots\xi_{n-1}e_v^r\otimes\mathrm{cod}(e_v^r)+$$
$$+\sum_{p=1}^{n}(-1)^{n-p-1}\xi_1\cdots\widetilde{f_p}\cdots\xi_{n-1}e_v^r\mathrm{cod}(e_v^r)\otimes 1 = \varphi_{n+1}+\psi_{n+1},$$

where

$$\psi_{n+1} = \sum_{p=1}^{n}(-1)^{n-p-1}\xi_1\cdots\widetilde{f_p}\cdots\xi_{n-1}e_v^r\mathrm{cod}(e_v^r)\otimes 1. \tag{13}$$

Hence, $\max\deg\psi_{n+1} = n$.

By (7), (9) and Lemma 1.1, we infer

$$i_n(\omega) = \sum_{j=1}^{n}(-1)^{n-j-1}\xi_1\cdots\widetilde{f_j}\cdots\xi_{n-1}e_v^r e_v^{r*}e_v^r\otimes 1 + \xi_1\cdots\widetilde{f_j}\cdots\xi_{n-1}e_v^r\mathrm{cod}(e_v^r)\otimes 1 + i_n(\omega') =$$
$$= \sum_{j=1}^{n}(-1)^{n-j-1}\xi_1\cdots\widetilde{f_j}\cdots\xi_{n-1}e_v^r e_v^{r*}e_v^r\otimes 1 + (-1)^{n-(n+1)-1}\xi_1\cdots\xi_{n-1}e_v^r\widetilde{f_{n+1}}(e_v^{r*}e^r)\otimes 1 + i_n(\omega') =$$
$$= -\sum_{j=1}^{n+1}(-1)^{n-j}\xi_1\cdots\widetilde{f_j}\cdots\xi_{n-1}e_v^r e_v^{r*}e_v^r\otimes 1 + i_n(\omega'). \tag{14}$$

Combining Lemma 1.1, (8), and (13), we infer

$$\omega' = -\sum_{j=1}^{n}\psi_j - \psi_{n+1} = -\sum_{p=1}^{n}\sum_{j=1}^{n}(-1)^{n-p-1}(-1)^{n-j-1}\xi_1\cdots\widetilde{f_p}\cdots\widetilde{f_j}\cdots\xi_{n-1}e_v^r e_v^{r*}e_v^r\otimes 1-$$
$$-\sum_{p=1}^{n}(-1)^{n-p-1}\xi_1\cdots\widetilde{f_p}\cdots\xi_{n-1}e_v^r\mathrm{cod}(e_v^r)\otimes 1 =$$
$$= -\sum_{p=1}^{n}\sum_{j=1}^{n}(-1)^{n-p-1}(-1)^{n-j-1}\xi_1\cdots\widetilde{f_p}\cdots\widetilde{f_j}\cdots\xi_{n-1}e_v^r e_v^{r*}e_v^r\otimes 1-$$
$$-(-1)^{n-(n+1)-1}\sum_{p=1}^{n}(-1)^{n-p-1}\xi_1\cdots\widetilde{f_p}\cdots\xi_{n-1}e_v^r\widetilde{f_{n+1}}(e_v^{r*}e_v^r)\otimes 1 =$$
$$= -\sum_{p=1}^{n}\sum_{j=1}^{n+1}(-1)^{n-p-1}(-1)^{n-j-1}\xi_1\cdots\widetilde{f_p}\cdots\widetilde{f_j}\cdots\xi_{n-1}e_v^r e_v^{r*}e_v^r\otimes 1.$$

Lemma 2.1 now yields $\omega' = 0$.

2. Suppose that $\xi_n \neq e_v$ and $\xi_{n+1} \neq e_v^*$. Then

$$\omega = \xi_1 \cdots \xi_n \otimes f(\xi_{n+1}\xi_{n+2}) + \sum_{j=1}^{n} (-1)^{n-j-1} \xi_1 \cdots \widetilde{f_j} \cdots \xi_n \xi_{n+1} \otimes \xi_{n+2}.$$

We have

$$f(\xi_{n+1}\xi_{n+2}) = \begin{cases} \mathrm{dom}(e_v) - \sum\limits_{r=2}^{\ell} e_v^r e_v^{r*}, \text{ iff } \xi_{n+1} = e_v,\, \xi_{n+2} = e_v^*, \\ \xi \in \{0\} \cup V \cup E \cup E^*, \text{ otherwise }. \end{cases}$$

(a) Consider the case where $f(\xi_{n+1}\xi_{n+2}) = \mathrm{dom}(e_v) - \sum\limits_{r=2}^{\ell} e_v^r e_v^{r*}$, i.e. $\xi_{n+1} = e_v$, $\xi_{n+2} = e_v^*$. We get

$$\omega = \xi_1 \cdots \xi_n \otimes \left(\mathrm{dom}(e_v) - \sum_{r=2}^{\ell} e_v^r e_v^{r*} \right) + \sum_{j=1}^{n} (-1)^{n-j-1} \xi_1 \cdots \widetilde{f_j} \cdots \xi_n e_v \otimes e_v^*.$$

By the length of words, we see that $\mathrm{HT}(\omega) = \xi_1 \cdots \xi_n e_v^2 e_v^{2*}$. Therefore, $h_n \mathrm{HT}(\omega) = \xi_1 \cdots \xi_n e_v^2 \otimes e_v^{2*}$ and by (6) we get

$$d_n h_n \mathrm{HT}(\xi_1 \cdots \xi_n e_v^2 \otimes e_v^{2*}) = \xi_1 \cdots \xi_n \otimes f(e_v^2 e_v^{2*}) + \sum_{j=1}^{n} (-1)^{n-j-1} \xi_1 \cdots \widetilde{f_j} \cdots \xi_n e_v^2 \otimes e_v^{2*},$$

then

$$\omega^{(2)} = \omega + d_n h_n \mathrm{HT}(\omega) = \xi_1 \cdots \xi_n \otimes \left(\mathrm{dom}(e_v) - \sum_{r=3}^{\ell} e_v^r e_v^{r*} \right) + \sum_{j=1}^{n} (-1)^{n-j-1} \xi_1 \cdots \widetilde{f_j} \cdots \xi_n e_v \otimes e_v^* +$$

$$+ \sum_{j=1}^{n} (-1)^{n-j-1} \xi_1 \cdots \widetilde{f_j} \cdots \xi_n e_v^2 \otimes e_v^{2*}.$$

It is not hard to see that $\mathrm{HT}(\omega^{(2)}) = \xi_1 \cdots \xi_n e_v^3 e_v^{3*}$ etc. So, after $\ell - 1$ steps we get

$$i_n(\omega) = -\sum_{r=2}^{\ell} \xi_1 \cdots \xi_n e_v^r \otimes e_v^{r*} + i_n(\omega^{(\ell)}), \tag{15}$$

where

$$\omega^{(\ell)} = \xi_1 \cdots \xi_n \otimes \mathrm{dom}(e_v) + \sum_{r=1}^{\ell} \sum_{j=1}^{n} (-1)^{n-j-1} \xi_1 \cdots \widetilde{f_j} \cdots \xi_n e_v^r \otimes e_v^{r*} =$$

$$= \sum_{j=1}^{n} (-1)^{n-j-1} \varphi_j + \varphi_{n+1}$$

Check that $\omega^{(\ell)}$ satisfies the conditions of the Lemma 1.1. For φ_{n+1} we have

$$h_n \mathrm{HT}(\varphi_{n+1}) = \xi_1 \cdots \xi_n \mathrm{dom}(e_v) \otimes 1 \tag{16}$$

and by (6) we obtain

$$d_n h_n \mathrm{HT}(\varphi_{n+1}) = d_n(\xi_1 \cdots \xi_n \mathrm{dom}(e_v) \otimes 1) = \xi_1 \cdots \xi_n \otimes \mathrm{dom}(e_v) +$$

$$+ \sum_{j=1}^{n} (-1)^{n-j-1} \xi_1 \cdots \widetilde{f_j} \cdots \xi_n \mathrm{dom}(e_v) \otimes 1 = \varphi_{n+1} + \psi_{n+1};$$

Moreover,

$$\psi_{n+1} = \sum_{j=1}^{n} (-1)^{n-j-1} \xi_1 \cdots \widetilde{f_j} \cdots \xi_n \mathrm{dom}(e_v) \otimes 1 \tag{17}$$

and we conclude that $\max \deg \psi_1 = n$. Further, for any $1 \leq j \leq n-1$ we have

$$h_n \mathrm{HT} \varphi_j = \xi_1 \cdots \widetilde{f_j} \cdots \xi_n e_v e_v^* \otimes 1 \tag{18}$$

and by (5) we obtain

$$(-1)^{n-j-1} d_n h_n \mathrm{HT} \varphi_j = (-1)^{n-j-1} d_n(\xi_1 \cdots \widetilde{f_j} \cdots \xi_n e_v e_v^* \otimes 1) =$$

$$= (-1)^{n-j-1} \sum_{r=1}^{\ell} \xi_1 \cdots \widetilde{f_j} \cdots \xi_n e_v^r \otimes e_v^{r*} +$$

$$+ (-1)^{n-j-1} \sum_{p=1}^{n} (-1)^{n-p-1} \xi_1 \cdots \widetilde{f_j} \cdots \widetilde{f_p} \cdots \xi_n e_v e_v^* \otimes 1 = \varphi_j + \psi_j.$$

Here

$$\psi_j = (-1)^{n-j-1}\sum_{p=1}^{n}(-1)^{n-p-1}\xi_1\cdots\widetilde{f_j}\cdots\widetilde{f_p}\cdots\xi_n e_v e_v^*\otimes 1. \tag{19}$$

We see that $\max\deg\psi_j = n$. For $j=n$, we have $\varphi_n = -\sum_{r=1}^{\ell}\xi\cdots\xi_{n-1}\widetilde{f}(\xi_n e_v^r)\otimes e_v^{r*}$, and so

$$\varphi_n = \begin{cases} -\sum\limits_{r=1}^{\ell}\xi_1\cdots\xi_{n-1}e_v^r\otimes e_v^{r*} \text{ if } \xi_n = \mathrm{dom}(e_v),\\ -\xi_1\cdots\xi_{n-1}\widetilde{f_n}\otimes e_v^{r*} \text{ where } \widetilde{f_n}\in\{0\}\cup V\cup E\setminus\{e_v^r\}\cup E^* \text{ for the other } \xi_n \text{ and for some } 1\le r\le\ell.\end{cases}$$

(i) Let $\varphi_n = -\sum_{r=1}^{\ell}\xi_1\cdots\xi_{n-1}e_v^r\otimes e_v^{r*}$. Then

$$h_n\mathrm{HT}(\varphi_n) = \xi_1\cdots\xi_{n-1}e_v e_v^*\otimes 1 \tag{20}$$

and hence

$$\begin{aligned} -\, d_n h_n\mathrm{HT}(\varphi_n) &= -d_n(\xi_1\cdots\xi_{n-1}e_v e_v^*\otimes 1) = \\ &= -\sum_{r=1}^{\ell}\xi_1\cdots\xi_{n-1}e_v^r\otimes e_v^{r*} - \sum_{p=1}^{n}(-1)^{n-p-1}\xi_1\cdots\widetilde{f_p}\cdots\xi_{n-1}e_v e_v^*\otimes 1 = \\ &= \varphi_n + \psi_n. \end{aligned}$$

Here

$$\psi_n = -\sum_{p=1}^{n}(-1)^{n-p-1}\xi_1\cdots\widetilde{f_p}\cdots\xi_{n-1}e_v e_v^*\otimes 1 \tag{21}$$

and $\max\deg\psi_n = n$.

Combining (16), (18), and (20), we obtain

$$\begin{aligned} i_n(\omega) = -\sum_{r=2}^{\ell}\xi_1\cdots\xi_n e_v^r\otimes e_v^{r*} + \xi_1\cdots\xi_n\mathrm{dom}(e_v)\otimes 1 + \sum_{j=1}^{n-1}(-1)^{n-j-1}\xi_1\cdots\widetilde{f_j}\cdots\xi_n e_v e_v^*\otimes 1 - \\ -\, \xi_1\cdots\xi_{n-1}e_v e_v^*\otimes 1 + i_n(\omega') \end{aligned}$$

and since $\widetilde{f_{n+1}}(e_v e_v^*) = \mathrm{dom}(e_v)$ and we have assumed that $\widetilde{f_n}(\xi_n e_v^r) = e_v^r$, we have

$$i_n(\omega) = -\sum_{r=2}^{\ell}\xi_1\cdots\xi_n e_v^r\otimes e_v^{r*} - \sum_{j=1}^{n+1}(-1)^{n-j}\xi_1\cdots\widetilde{f_j}\cdots\xi_n e_v e_v^*\otimes 1 + i_n(\omega') \tag{22}$$

where, by (17), (19) and (21), we infer

$$\begin{aligned} \omega' = -\psi_{n+1} - \sum_{j=1}^{n-1}\psi_j - \psi_n &= -\sum_{j=1}^{n}(-1)^{n-j-1}\xi_1\cdots\widetilde{f_j}\cdots\xi_n\mathrm{dom}(e_v)\otimes 1 - \\ &\quad -\sum_{j=1}^{n-1}\sum_{p=1}^{n}(-1)^{n-j-1}(-1)^{n-p-1}\xi_1\cdots\widetilde{f_j}\cdots\widetilde{f_p}\cdots\xi_n e_v e_v^*\otimes 1 + \\ &\quad +\sum_{p=1}^{n}(-1)^{n-p-1}\xi_1\cdots\widetilde{f_p}\cdots\xi_{n-1}e_v e_v^*\otimes 1 = \\ &= -(-1)^{n-(n+1)-1}\sum_{j=1}^{n}(-1)^{n-j-1}\xi_1\cdots\widetilde{f_j}\cdots\xi_n\widetilde{f_{n+1}}(e_v e_v^*)\otimes 1 - \\ &\quad -\sum_{j=1}^{n-1}\sum_{p=1}^{n}(-1)^{n-j-1}(-1)^{n-p-1}\xi_1\cdots\widetilde{f_j}\cdots\widetilde{f_p}\cdots\xi_n e_v e_v^*\otimes 1 - \\ &\quad -(-1)^{n-n-1}\sum_{p=1}^{n}(-1)^{n-p-1}\xi_1\cdots\widetilde{f_p}\cdots\xi_{n-1}\widetilde{f_n}(\xi_n e_v)e_v^*\otimes 1 = \\ &= -(-1)^{n-(n+1)-1}\sum_{j=1}^{n}(-1)^{n-j-1}\xi_1\cdots\widetilde{f_j}\cdots\xi_n\widetilde{f_{n+1}}(e_v e_v^*)\otimes 1 - \\ &\quad -\sum_{j=1}^{n}\sum_{p=1}^{n}(-1)^{n-j-1}(-1)^{n-p-1}\xi_1\cdots\widetilde{f_j}\cdots\widetilde{f_p}\cdots\xi_n e_v e_v^*\otimes 1 = \\ &= -\sum_{j=1}^{n}\sum_{p=1}^{n-1}(-1)^{n-j-1}(-1)^{n-p-1}\xi_1\cdots\widetilde{f_j}\cdots\widetilde{f_p}\cdots\xi_n e_v e_v^*\otimes 1. \end{aligned}$$

Lemma 2.1 implies that $\omega' = 0$.

(ii) Suppose that $\varphi_n = -\xi_1 \cdots \xi_{n-1}\widetilde{f}_n \otimes e_v^{r*}$ for some $1 \le r \le \ell$. Then

$$h_n\mathrm{HT}(\varphi_n) = \xi_1 \cdots \xi_{n-1}\widetilde{f}_n e_v^{r*} \otimes 1 \tag{23}$$

By (6), we obtain

$$d_nh_n\mathrm{HT}(\varphi_n) = \xi_1 \cdots \xi_{n-1}\widetilde{f}_n e_v^{r*} \otimes 1 + \sum_{p=1}^{n}(-1)^{n-p-1}\xi_1 \cdots \widetilde{f}_p \cdots \xi_{n-1}\widetilde{f}_n e_v^{r*} \otimes 1 = \varphi_n + \psi_n,$$

where

$$\psi_n = \sum_{p=1}^{n}(-1)^{n-p-1}\xi_1 \cdots \widetilde{f}_p \cdots \xi_{n-1}\widetilde{f}_n e_v^{r*} \otimes 1 \tag{24}$$

and we see that $\max\deg\psi_n = n$.

Therefore,

$$i_n(\omega) = -\sum_{r=2}^{\ell}\xi_1 \cdots \xi_n e_v^r \otimes e_v^{r*} + \xi_1 \cdots \xi_n \mathrm{dom}(e_v) \otimes 1 + \sum_{j=1}^{n-1}(-1)^{n-j-1}\xi_1 \cdots \widetilde{f}_j \cdots \xi_n e_v e_v^* \otimes 1 - \\ - \xi_1 \cdots \xi_{n-1}\widetilde{f_n} e_v^{r*} \otimes 1 + i_n(\omega')$$

but, reckoning with the relations $\widetilde{f_n}(\xi e_v^r) = \widetilde{f_n}$ and $\widetilde{f_{n+1}}(e_v e_v^*) = \mathrm{dom}(e_v)$, we have

$$i_n(\omega) = -\sum_{r=2}^{\ell}\xi_1 \cdots \xi_n e_v^r \otimes e_v^{r*} - \sum_{j=1}^{n+1}(-1)^{n-j}\xi_1 \cdots \widetilde{f}_j \cdots \xi_n e_v e_v^* \otimes 1 + i_n(\omega'). \tag{25}$$

Combining (17), (19), and (24), we get

$$\begin{aligned}
\omega' = -\psi_{n+1} - \sum_{j=1}^{n-1}\psi_j - \psi_n = &-\sum_{j=1}^{n}(-1)^{n-j-1}\xi_1 \cdots \widetilde{f}_j \cdots \xi_n \mathrm{dom}(e_v) \otimes 1 - \\
&- \sum_{j=1}^{n-1}\sum_{p=1}^{n}(-1)^{n-j-1}(-1)^{n-p-1}\xi_1 \cdots \widetilde{f}_j \cdots \widetilde{f}_p \cdots \xi_n e_v e_v^* \otimes 1 + \\
&+ \sum_{p=1}^{n}(-1)^{n-p-1}\xi_1 \cdots \widetilde{f}_p \cdots \xi_{n-1}\widetilde{f}_n e_v^{r*} \otimes 1 = \\
= &-(-1)^{n-(n+1)-1}\sum_{j=1}^{n}(-1)^{n-j-1}\xi_1 \cdots \widetilde{f}_j \cdots \xi_n \widetilde{f_{n+1}}(e_v e_v^*) \otimes 1 - \\
&- \sum_{j=1}^{n-1}\sum_{p=1}^{n}(-1)^{n-j-1}(-1)^{n-p-1}\xi_1 \cdots \widetilde{f}_j \cdots \widetilde{f}_p \cdots \xi_n e_v e_v^* \otimes 1 - \\
&- (-1)^{n-n-1}\sum_{p=1}^{n}(-1)^{n-p-1}\xi_1 \cdots \widetilde{f}_p \cdots \xi_{n-1}\widetilde{f}_n e_v^{r*} \otimes 1 = \\
= &-(-1)^{n-(n+1)-1}\sum_{j=1}^{n}(-1)^{n-j-1}\xi_1 \cdots \widetilde{f}_j \cdots \xi_n \widetilde{f_{n+1}}(e_v e_v^*) \otimes 1 - \\
&- \sum_{j=1}^{n}\sum_{p=1}^{n}(-1)^{n-j-1}(-1)^{n-p-1}\xi_1 \cdots \widetilde{f}_j \cdots \widetilde{f}_p \cdots \xi_n e_v e_v^* \otimes 1 = \\
&= -\sum_{j=1}^{n}\sum_{p=1}^{n+1}(-1)^{n-j-1}(-1)^{n-p-1}\xi_1 \cdots \widetilde{f}_j \cdots \widetilde{f}_p \cdots \xi_n e_v e_v^* \otimes 1.
\end{aligned}$$

Lemma 2.1 implies that $\omega' = 0$.

(b) Suppose that $\xi_{n+1} \ne e_v$, $\xi_{n+2} \ne e_v^*$, and $f(\xi_{n+1}\xi_{n+2}) = \xi$. Then

$$\omega = \xi_1 \cdots \xi_n \otimes \xi + \sum_{j=1}^{n}(-1)^{n-j-1}\xi_1 \cdots \widetilde{f}_j \cdots \xi_n\xi_{n+1} \otimes \xi_{n+2} = \sum_{j=1}^{n+1}(-1)^{n-j-1}\varphi_j.$$

Here, for any $1 \le j \le n$, we have $\varphi_j = \xi_1 \cdots \widetilde{f}_j \cdots \xi_{n+1} \otimes \xi_{n+2}$ and $\varphi_{n+1} = \xi_1 \cdots \xi_n \otimes \xi$; consequently,

$$h_n\mathrm{HT}(\varphi_j) = \xi_1 \cdots \widetilde{f}_j \cdots \xi_{n+2} \otimes 1, \qquad 1 \le j \le n \tag{26}$$

and

$$h_n\mathrm{HT}(\varphi_{n+1}) = \xi_1 \cdots \xi_n\xi \otimes 1 \tag{27}$$

By (6), we obtain

$$(-1)^{n-j-1}d_nh_n\mathrm{HT}(\varphi_j) = (-1)^{n-j-1}\xi_1\cdots\widetilde{f_j}\cdots\xi_{n+1}\otimes\xi_{n+2}+$$
$$+(-1)^{n-j-1}\sum_{p=1}^{n}(-1)^{n-p-1}\xi_1\cdots\widetilde{f_p}\cdots\widetilde{f_j}\cdots\xi_{n+1}\xi_{n+2}\otimes 1 = \varphi_j+\psi_j.$$

Here

$$\psi_j = (-1)^{n-j-1}\sum_{p=1}^{n}(-1)^{n-p-1}\xi_1\cdots\widetilde{f_p}\cdots\widetilde{f_j}\cdots\xi_{n+1}\xi_{n+2}\otimes 1 \tag{28}$$

We see that $\max\deg\psi_n = n$. Moreover, for φ_{n+1} by (6) we get

$$d_nh_n\mathrm{HT}(\varphi_{n+1}) = \xi_1\cdots\xi_n\otimes\xi+\sum_{p=1}^{n}(-1)^{n-p-1}\xi_1\cdots\widetilde{f_p}\cdots\xi_n\xi\otimes 1 = \varphi_{n+1}+\psi_{n+1}.$$

Here

$$\psi_{n+1} = \sum_{p=1}^{n}(-1)^{n-p-1}\xi_1\cdots\widetilde{f_p}\cdots\xi_n\xi\otimes 1 \tag{29}$$

and $\max\deg\psi_{n+1} = n$.

Combining (26) and (27), we infer

$$i_n(\omega) = \sum_{j=1}^{n}(-1)^{n-j-1}\xi_1\cdots\widetilde{f_j}\cdots\xi_{n+2}\otimes 1+\xi_1\cdots\xi_n\xi\otimes 1+i_n(\omega') =$$
$$=\sum_{j=1}^{n}(-1)^{n-j-1}\xi_1\cdots\widetilde{f_j}\cdots\xi_{n+2}\otimes 1+(-1)^{n-(n+1)-1}\xi_1\cdots\xi_n\xi\otimes 1+i_n(\omega') =$$
$$=\sum_{j=1}^{n+1}(-1)^{n-j-1}\xi_1\cdots\widetilde{f_j}\cdots\xi_{n+2}\otimes 1+i_n(\omega'). \tag{30}$$

Using (28) and (29), we get

$$\omega' = -\sum_{j=1}^{n}\psi_j-\psi_{n+1} = -\sum_{p=1}^{n}\sum_{j=1}^{n}(-1)^{n-j-1}(-1)^{n-p-1}\xi_1\cdots\widetilde{f_p}\cdots\widetilde{f_j}\cdots\xi_{n+1}\xi_{n+2}\otimes 1-$$
$$-\sum_{p=1}^{n}(-1)^{n-p-1}\xi_1\cdots\widetilde{f_p}\cdots\xi_n\xi\otimes 1 =$$
$$-\sum_{p=1}^{n}\sum_{j=1}^{n}(-1)^{n-j-1}(-1)^{n-p-1}\xi_1\cdots\widetilde{f_p}\cdots\widetilde{f_j}\cdots\xi_{n+1}\xi_{n+2}\otimes 1-$$
$$-(-1)^{n-(n+1)-1}\sum_{p=1}^{n}(-1)^{n-p-1}\xi_1\cdots\widetilde{f_p}\cdots\xi_n\xi\otimes 1 =$$
$$=-\sum_{p=1}^{n}\sum_{j=1}^{n+1}(-1)^{n-j-1}(-1)^{n-p-1}\xi_1\cdots\widetilde{f_p}\cdots\widetilde{f_j}\cdots\xi_{n+1}\xi_{n+2}\otimes 1.$$

Lemma 2.1 implies that $\omega' = 0$.

Thus, we have considered all cases. Combining (11) and (14), we finally obtain in the case of $\xi_{n+1} = e_v$ and $\xi_{n+2} = e_v^*$:

$$i_nd_n(\xi_1\cdots\xi_ne_v\otimes e_v) = -\sum_{r=2}^{\ell}\xi_1\cdots\xi_{n-1}e_v^r\otimes e_v^{r*}-\sum_{j=1}^{n+1}(-1)^{n-j}\xi_1\cdots\widetilde{f_j}\cdots\xi_{n-1}e_ve_v^*\otimes 1, \tag{31}$$

and, for $\xi_{n+1}\neq e_v$, $\xi_{n+2}\neq e_v^*$, we get by (22), (25) and (30):

$$i_nd_n(\xi_1\cdots\xi_{n+1}\otimes\xi_{n+2}) = -\sum_{j=1}^{n+1}(-1)^{n-j}\xi_1\cdots\widetilde{f_j}\cdots\xi_{n+2}\otimes 1. \tag{32}$$

Since $d_{n+1}(\xi_1\cdots\xi_{n+2}\otimes 1) = \xi_1\cdots\xi_{n+1}\otimes\xi_{n+2}-i_nd_n(\xi_1\cdots\xi_{n+1}\otimes\xi_{n+2})$, we appeal to (31) and (32) for finishing the proof.

□

REMARK 2.1 *A Leavitt path algebra satisfies condition (1.3). Indeed, the proof of Theorem 2.1 implies that* $\mathrm{HT}(\omega - h_n \mathrm{HT}_c(\omega)) \geq \mathrm{HT}(d_n h_n \mathrm{HT}(\omega) - h_n \mathrm{HT}_c d_n h_n \mathrm{HT}(\omega))$, *for each* $n \geq 0$ *and any* $\omega \in \ker(d_{n-1})$.

THEOREM 2.2. *For a Leavitt path algebra, we have*

$$\mathrm{Tor}_n^{L(\Gamma)}(\Bbbk, \Bbbk) = 0, \qquad \textit{for all } n > 0$$

PROOF. Since in general we assume that $\varepsilon(\xi) = 0$ for any generator of the Leavitt path algebra, after tensoring the resolution from Theorem 2.1 with $\Bbbk$ over $L(\Gamma)$, we obtain the chain complex

$$0 \leftarrow \Bbbk \xleftarrow{0} \mathrm{Span}_\Bbbk(V \cup E \cup E^*) \xleftarrow{\bar{d}_1} \mathrm{Span}_\Bbbk \mathfrak{V}_{L(\Gamma)} \xleftarrow{\bar{d}_2} \mathrm{Span}_\Bbbk \mathfrak{V}^{(2)}_{L(\Gamma)} \xleftarrow{\bar{d}_3} \mathrm{Span}_\Bbbk \mathfrak{V}^{(3)}_{L(\Gamma)} \leftarrow \dots.$$

Here the differentials are expressed by the formulas

$$\bar{d}_n(\xi_1 \cdots \xi_{n+1} \otimes 1) = \sum_{j=1}^{n} (-1)^{n-j-1} \xi_1 \cdots \widetilde{f}_j \cdots \xi_{j+1}.$$

Note that a Leavitt path algebra satisfies the following condition: for any generator ξ, there exists a (not necessarily unique) generator ζ such that $\widetilde{f}(\xi\zeta) = \xi$. Indeed, choose the same $\zeta = v$ for every vertex $\xi = v \in V$, take $\zeta = \mathrm{cod}(e)$ for every edge $\xi = e \in E$, and put $\zeta = \mathrm{dom}(e)$ and for any $\xi = e^* \in E^*$. Let $\xi_1 \cdots \xi_{n-1} \in \mathfrak{V}^{(n)}$ be an n-chain. Given fixed $\zeta \in V \cup E \cup E^*$ with $\widetilde{f}(\xi_{n+1}\zeta) = \xi_{n+1}$, introduce a map $\varrho^\zeta_{n+1} : \mathfrak{V}^{(n)} \to \mathfrak{V}^{(n+1)}$ by $\varrho^\zeta_{n+1} = -\xi_1 \cdots \xi_{n+1}\zeta$. Show that this is a contracting homotopy; i.e.,

$$\bar{d}_{n+1}\varrho^\zeta_{n+1} + \varrho^\zeta_n \bar{d}_n = \mathrm{id}_{\mathfrak{V}^{(n)}\Bbbk}$$

Indeed,

$$\varrho^\zeta_n \bar{d}_n(\xi_1 \cdots \xi_{n+1}) = \varrho^\zeta_n \left(\sum_{j=1}^{n} (-1)^{n-j-1} \xi_1 \cdots \widetilde{f}_j \cdots \xi_{n+1} \right) =$$

$$= - \sum_{j=1}^{n} (-1)^{n-j-1} \xi_1 \cdots \widetilde{f}_j \cdots \xi_{n+1}\zeta.$$

On the other hand,

$$\bar{d}_{n+1}\varrho^\zeta_{n+1}(\xi_1 \cdots \xi_{n+1}) = -\bar{d}_{n+1}(\xi_1 \cdots \xi_{n+1}\zeta) = -\sum_{j=1}^{n+1} (-1)^{n-j} \xi_1 \cdots \widetilde{f}_j \cdots \xi_{n+1}\zeta =$$

$$= \xi_1 \cdots \xi_{n+1} + \sum_{j=1}^{n} (-1)^{n-j-1} \xi_1 \cdots \widetilde{f}_j \cdots \xi_{n+1}\zeta,$$

and we see that $\bar{d}_{n+1}\varrho^\zeta_{n+1}(\xi_1 \cdots \xi_{n+1}) + \varrho^\zeta_n \bar{d}_n(\xi_1 \cdots \xi_{n+1}) = \xi_1 \cdots \xi_{n+1}$, as claimed. This means that the complex for the Leavitt path algebra is acyclic; i.e., $\mathrm{Tor}_n^{L(\Gamma)}(\Bbbk, \Bbbk) = 0$ for $n > 0$.

□

2.1 Anick's Resolution and the Ring of Laurent Series.

Here we consider a special case of a Leavitt algebra. If we take a directed graph of the form $\Omega = (V, E)$, where $V = \{v\}$ and $E = \{e\}$ then $L(\Omega) \cong \Bbbk[t, t^{-1}]$. Indeed, we have the relations

$$v^2 = v,\ ve = ev = e,\ ve^* = e^*v = e^*,\ ee^* = e^*e = v.$$

Using Propositions 2.1 and 2.2, describe the set of Anick n-chains as the set of strings $\{\zeta_1 \cdots \zeta_{n+1}\}$, where $\zeta \in \{v, e, e^*\}$ and $\zeta_i \neq \zeta_{i+1}$ iff $\zeta_i \neq v$, and so $\mathfrak{V}^{(n)}_{L(\Omega)}\Bbbk = \bigoplus_{\zeta_{\iota_1}, \dots, \zeta_{\iota_{n+1}} \in \{v, e, e^*\}} \mathrm{Span}_\Bbbk(\{\zeta_{\iota_1} \cdots \zeta_{\iota_{n+1}}\})$. As an augmentation $\varepsilon : L(\Omega) \to \Bbbk$, we take $\varepsilon(v) = \varepsilon(e) = \varepsilon(e^*) = 1$. In this case, we get another Anick's resolution.

THEOREM 2.3. *If* Ω *is a directed graph with a single vertex and a single edge then Anick's resolution for the* $L(\Omega)$*-module* $\Bbbk$ *is given by the (exact) chain complex*

$$0 \leftarrow \Bbbk \xleftarrow{\varepsilon} L(\Omega) \xleftarrow{d_0} \mathrm{Span}_\Bbbk(v, e, e^*) \otimes_\Bbbk L(\Omega) \xleftarrow{d_1} \mathfrak{V}^{(1)}_{L(\Omega)}\Bbbk \otimes_\Bbbk L(\Omega) \xleftarrow{d_2} \mathfrak{V}^{(2)}_{L(\Omega)}\Bbbk \otimes_\Bbbk L(\Omega) \xleftarrow{d_3} \dots.$$

Here $\zeta_{i_k} \in V \cup E \cup E^*$ *for each* $k > 0$ *and*

$$\varepsilon(v) = \varepsilon(e) = \varepsilon(e^*) = 1, \quad d_0(v \otimes 1) = v - 1, \quad d_0(e \otimes 1) = e - 1, \quad d_0(e^* \otimes 1) = e^* - 1$$

for each $n > 0$:

$$d_n(\zeta_1 \cdots \zeta_{n+1} \otimes 1) = \zeta_1 \cdots \zeta_n \otimes \zeta_{n+1} + \sum_{t=0}^{n} (-1)^{n-t-1} \zeta_1 \cdots \widetilde{f_t} \cdots \zeta_{n+1} \otimes 1, \tag{33}$$

where $\widetilde{f_t} = f(\zeta_t \zeta_{t+1})$, and $\widetilde{f_0}$ means removing ζ_1 from $\zeta_1 \cdots \zeta_{n+2}$.

PROOF. Use induction on n. Suppose that (33) holds for some $n \geq 0$. We have

$$d_{n+1}(\zeta_1 \cdots \zeta_{n+2} \otimes 1) = \zeta_1 \cdots \zeta_{n+1} \otimes \zeta_{n+2} - i_n d_n(\zeta_1 \cdots \zeta_{n+1} \otimes \zeta_{n+2}).$$

Using (33) we get

$$d_n(\zeta_1 \cdots \zeta_{n+1} \otimes \zeta_{n+2}) = \zeta_1 \cdots \zeta_n \otimes \widetilde{f_{n+1}} + \sum_{t=0}^{n} (-1)^{n-t-1} \zeta_1 \cdots \widetilde{f_t} \cdots \zeta_{n+1} \otimes \zeta_{n+2}.$$

Let us check the conditions of Lemma 1.1. Let $\omega = \varphi_1 + \sum_{t=0}^{n} (-1)^{n-t-1} \varphi_t$, where $\varphi_1 = \zeta_1 \cdots \zeta_n \otimes \widetilde{f_{n+1}}$ and $\varphi_j = \zeta_1 \cdots \widetilde{f_t} \cdots \zeta_{n+1} \otimes \zeta_{n+2}$. We have $\mathrm{HT}(\varphi_1) = \zeta_1 \cdots \zeta_n \widetilde{f_{n+1}}$, and hence $h_n \mathrm{HT}(\varphi_1) = \zeta_1 \cdots \zeta_n \widetilde{f_{n+1}} \otimes 1$. Now, by (33), we infer

$$d_n h_n \mathrm{HT} \varphi_1 = \zeta_1 \cdots \zeta_n \otimes \widetilde{f_{n+1}} + \sum_{t=0}^{n} (-1)^{n-t-1} \zeta_1 \cdots \widetilde{f_t} \cdots \zeta_n \widetilde{f_{n+1}} \otimes 1 = \varphi_1 + \psi_1,$$

where

$$\psi_1 = \sum_{t=0}^{n} (-1)^{n-t-1} \zeta_1 \cdots \widetilde{f_t} \cdots \zeta_n \widetilde{f_{n+1}} \otimes 1$$

and we see that $\max \deg \psi_1 = n$. Further, for fixed $0 \leq s \leq n$ we have $\mathrm{HT}(\varphi_s) = \zeta_1 \cdots \widetilde{f_s} \cdots \zeta_{n+1} \zeta_{n+2}$ and $h_n \mathrm{HT}(\varphi_s) = \zeta_1 \cdots \widetilde{f_s} \cdots \zeta_{n+1} \zeta_{n+2} \otimes 1$. Then (33) yields

$$\begin{aligned}(-1)^{n-s-1} d_n h_n \mathrm{HT}(\varphi_s) &= (-1)^{n-s-1} \zeta_1 \cdots \widetilde{f_s} \cdots \zeta_{n+1} \otimes \zeta_{n+2} + \\ &+ (-1)^{n-s-1} \sum_{t=0}^{n} (-1)^{n-t-1} \zeta_1 \cdots \widetilde{f_t} \cdots \widetilde{f_s} \cdots \zeta_{n+2} \otimes 1 = \\ &= \varphi_s + \psi_s,\end{aligned}$$

where

$$\psi_s = (-1)^{n-s-1} \sum_{t=0}^{n} (-1)^{n-t-1} \zeta_1 \cdots \widetilde{f_t} \cdots \widetilde{f_s} \cdots \zeta_{n+2} \otimes 1,$$

and we conclude that $\max \deg \psi_s = n$. Then Lemma 1.1 gives

$$\begin{aligned} i_n d_n(\zeta_1 \cdots \zeta_{n+1} \otimes \zeta_{n+2}) &= (-1)^{n-(n+1)-1} \zeta_1 \cdots \zeta_n \widetilde{f_{n+1}} \otimes 1 + \\ &+ \sum_{s=0}^{n} (-1)^{n-s-1} \zeta_1 \cdots \widetilde{f_s} \cdots \zeta_{n+2} \otimes 1 + i_n(\omega') = \\ &= -\sum_{s=0}^{n+1} (-1)^{n-s} \zeta_1 \cdots \widetilde{f_s} \cdots \zeta_{n+2} \otimes 1 + i_n(\omega'),\end{aligned}$$

where

$$\begin{aligned}\omega' = -\psi_1 - \sum_{s=0}^{n} \psi_s &= -(-1)^{n-(n+1)-1} \sum_{t=0}^{n} (-1)^{n-t-1} \zeta_1 \cdots \widetilde{f_t} \cdots \zeta_n \widetilde{f_{n+1}} \otimes 1 - \\ &- \sum_{s=0}^{n} \sum_{t=0}^{n} (-1)^{n-s-1} (-1)^{n-t-1} \zeta_1 \cdots \widetilde{f_t} \cdots \widetilde{f_s} \cdots \zeta_{n+2} \otimes 1 = \\ &- \sum_{s=0}^{n+1} \sum_{t=0}^{n} (-1)^{n-s-1} (-1)^{n-t-1} \zeta_1 \cdots \widetilde{f_t} \cdots \widetilde{f_s} \cdots \zeta_{n+2} \otimes 1\end{aligned}$$

Consider the sum $\sum_{s=0}^{n+1} \sum_{t=0}^{n} b_{st}$, where $b_{st} = (-1)^{n-s-1} (-1)^{n-t-1} \zeta_1 \cdots \widetilde{f_t} \cdots \widetilde{f_s} \cdots \zeta_{n+2}$. We infer that $b_{00} = \zeta_3 \cdots \zeta_{n+2} = -b_{01}$ and $b_{10} = f(\zeta_2 \zeta_3) \zeta_4 \cdots \zeta_{n+2} = -b_{02}$. Now, making use of the proof of Lemma 2.1, we finally get $\omega' = 0$. Thus,

$$d_{n+1}(\zeta_1 \cdots \zeta_{n+2} \otimes 1) = \zeta_1 \cdots \zeta_{n+1} \otimes \zeta_{n+2} + \sum_{s=0}^{n+1} (-1)^{n-s} \zeta_1 \cdots \widetilde{f_s} \cdots \zeta_{n+2} \otimes 1$$

q.e.d.

$\square$

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